Finite Population Unequal Probability Bayesian Bootstraps and Multiple Imputation

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Outline

- Efron’s Bootstrap
- Rubin’s Bayesian Bootstrap
- Gross’s Finite Population Bootstrap
- Lo’s Finite Population Bayesian Bootstrap
- Unequal Probability Bayesian Bootstrap
- Connections to Multiple Imputation
- Recent Work

- Observed values \( x = (x_1, x_2, \ldots, x_n) \)
- Bootstrap values \( x^* = (x_1^*, x_2^*, \ldots, x_n^*) \) sampled \( n \) times with replacement from \( x \).
- Do \( B \) bootstrap samples \( x^{*1}, x^{*2}, \ldots, x^{*B} \).
- \( \hat{\theta}(x) \), \( \hat{\theta}(x^{*b}) \)
- \( \hat{\theta}(x^{**}) = \sum_{b=1}^{B} \hat{\theta}(x^{*b}) / B \)
- \( \hat{\text{SE}}[\hat{\theta}(x)] = \left\{ \frac{1}{B-1} \sum_{b=1}^{B} [\hat{\theta}(x^{*b}) - \hat{\theta}(x^{**})]^2 \right\}^{\frac{1}{2}} \)
Rubin’s Bayesian Bootstrap (BB), Rubin (1979, 1982), Lo (1987)

- **Step 1.** Draw $n - 1$ uniform $[0, 1]$ r.v.’s. Let their ordered values be $a_1, a_2, \ldots, a_{n-1}$. Let $a_0 = 0$; $a_n = 1$.

- **Step 2.** Draw each of the $n$ values in $x^{*b} = (x_1^{*b}, x_2^{*b}, \ldots, x_n^{*b})$ independently from $x_1, x_2, \ldots, x_n$ with probabilities $(a_1 - a_0), (a_2 - a_1), \ldots, (a_n - a_{n-1})$. 
Rubin’s BB: Why Bayesian?

- Vector of probabilities $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$. 
  $x_i$ i.i.d., $\Pr(x_i = d_k) = \lambda_k$, $\sum \lambda_k = 1$.

- Rubin(1981) showed BB like assuming (improper) prior
  $\Pr(\lambda) = \prod_{k=1}^{K} \lambda_k^{-1}$ if $\sum \lambda_k = 1$ and 0 otherwise.

- Posterior:
  $\Pr(\lambda) \propto \prod_{k=1}^{K} \lambda_k^{n_k-1}$
  where $n_k = \# \{x_i = d_k\}$.

- Lo (1987) showed BB has same desirable large sample properties as Efron’s bootstrap.
Finite Population Bootstrap (FPB) of Gross (1980)

- Finite population bootstrap (FPB), Gross (1980)
- Let \( y = (y_1, y_2, \ldots, y_n) \) be sample from population \( Y = (Y_1, Y_2, \ldots, Y_N) \), \( n \leq N - 1 \)
- Simple random sampling, either with or without replacement
- Assume for simplicity \( N = kn \), integer \( k \).
- Create FBP population \( Y^* = (Y_1^*, Y_2^*, \ldots, Y_N^*) \) with \( k \) copies of sample.
- Each FPB sample \( y^* = (y_1^*, y_2^*, \ldots, y_n^*) \) is a simple random sample without replacement from \( Y^* \).
- See Chapter 6 of Shao and Tu (1995) for extensions.
Finite Population Bayesian Bootstrap (FPBB) of Lo (1988)

“Pólya Urn Scheme”

- An urn contains a finite number of balls.
- Select ball from urn at random
- Ball is replaced and another ball just like it is also added to urn.
- Continue until a fixed number, say $m$, of balls is selected.
- An urn containing $z_1, z_2, \ldots, z_n$ will be denoted by $\text{urn}\{z_1, z_2, \ldots, z_n\}$. 
Calculation of a FPBB Replicate

Each replication of FPBB is formed as follows (adapted from Lo, 1988, p. 1686):

- **Step 1.** Draw a Pólya sample of size $N - n$, denoted by $y_1^*, y_2^*, \ldots, y_{N-n}^*$ from $\text{urn}\{y_1, y_2, \ldots, y_n\}$.
- **Step 2.** Form the FPBB population $y_1, y_2, \ldots, y_n, y_1^*, y_2^*, \ldots, y_{N-n}^*$.

Lo’s FPBB resamples the population outside the sample rather than resampling the sample itself.
Unequal Probability Bayesian Bootstrap (UPBB)

Unequal Probability Sampling

- In survey sampling, it is common to select units with unequal probabilities. For example, if $x_i$ is a measure of size (say, number of employees or total revenue) of business establishment $i$, we might select $i$ into the sample with a probability proportional to $x_i$.
- Let $\pi_i$ be the probability that unit $i$ is selected into the sample.
- The (base) weight is $w_i = 1/\pi_i$.
- If $x_i$ is the number of employees at establishment $i$ and establishment $i$ is selected into the sample, then $w_i$ can be thought of as the number of employees in the population that establishment $i$ represents. ($\sum_S w_i x_i$ estimates number of employees in population.)
Unequal Probability Bayesian Bootstrap (UPBB), cont.

Calculation of a UPBB Replicate

Each replication of UPBB is formed in two steps (Cohen, 1997):

- **Step 1.** Draw a sample of size $N - n$, denoted by $y_1^*, y_2^*, \ldots, y_{N-n}^*$, as follows:
  
  Determine $y_k^*$ by drawing from $y_1, y_2, \ldots, y_n$ with probability
  
  $$
  w_i - 1 + \ell_{i,k-1} \frac{N-n}{n}
  
  N - n + (k + 1) \frac{N-n}{n}
  $$

  where $\ell_{i,k-1} = \text{number of bootstrap selections of } y_i \text{ among } y_1^*, y_2^*, \ldots, y_{k-1}^*$. Set $\ell_{i,0} = 0$ and note that
  
  $$
  \sum_{i=1}^{n} \ell_{i,k-1} = k - 1.
  $$

- **Step 2.** Form the UPBB population
  
  $y_1, y_2, \ldots, y_n, y_1^*, y_2^*, \ldots, y_{N-n}^*$. 

UPBB: Why is it Bayesian?

- Vector of probabilities \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K) \).
  \[
  \Pr(y_i = d_k) = \lambda_k, \quad \sum \lambda_k = 1.
  \]
- From Dong, Elliott, and Raghunathan (2014), assume (improper) prior
  \[
  \Pr(\lambda) = \prod_{k=1}^{K} \lambda_k^{-1} \quad \text{if} \quad \sum \lambda_k = 1 \quad \text{and} \quad 0 \quad \text{otherwise}.
  \]
- Let \( z_k = \sum_{j=1}^{n} (w_j - 1) \times I(y_j = d_k) \) and \( n_k = \# \{ y_i = d_k \} \).
- Likelihood:
  \[
  \Pr(y_1, \ldots, y_n | \lambda) \propto \prod_{k=1}^{K} \lambda_k^{z_k}.
  \]
- Posterior:
  \[
  \Pr(y_1^*, y_2^*, \ldots, y_{N-n}^* | y_1, y_2, \ldots, y_n) \propto \prod_{k=1}^{K} \frac{\Gamma(z_k+n_k)}{\Gamma(z_k)}
  \]
Lo’s FPBB and Multiple Imputation (MI)

- Goal of imputation is to produce a *complete sample*.
- Multiple imputation repeats the imputation process to assess variability.
- Correspondence between Lo’s FPBB and multiple imputation:

<table>
<thead>
<tr>
<th>Lo’s FPBB</th>
<th>Multiple Imputation</th>
</tr>
</thead>
<tbody>
<tr>
<td>population size $N$</td>
<td>sample size $n$</td>
</tr>
<tr>
<td>sample size $n$</td>
<td>number of respondents $r$</td>
</tr>
<tr>
<td>size of nonsample $N - n$</td>
<td>number of missing $m = n - r$</td>
</tr>
</tbody>
</table>
Multiple Imputation (MI)

- Let $\mathcal{I}_j$ denote the indicator that is 1 if unit $j$ was sampled \textbf{and} responded, 0 otherwise. Let $\mathcal{I}$ be the vector of $\mathcal{I}_j$ values, $j = 1, \ldots, n$.

- Let $c_{j}^{*b} = \#\{y_{i}^{*b} = y_{j}\}$ be the number of times respondent $j$ is used in bootstrap replicate $b$ ($c_{j}^{*b} \geq 1$). Let $c^{*}$ be the vector of $c_{j}^{*b}$ values for a specific $b$.

- Then

$$\text{var} \hat{\theta}(\mathcal{I}, c^{*}) = \text{var}_{\mathcal{I}} \mathbb{E}_{*}\left[\hat{\theta}(\mathcal{I}, c^{*})|\mathcal{I}\right] + \mathbb{E}_{\mathcal{I}} \text{var}_{*}\left[\hat{\theta}(\mathcal{I}, c^{*})|\mathcal{I}\right].$$
Dong, Elliott, and Raghunathan (2014)

- Gives a nonparametric method to generate synthetic populations accounting for complex sampling (not MI).
- Uses UPBB.
- Notes that draws from “weighted” Pólya urn can be produced using function wtpolyap in R package `polypost`.
Zhou, Elliott, and Raghunathan (2016)

- Treats MI.
- Uses UPBB.
- Considers two-stage designs (e.g., sampling schools, then students within the school).
- Zhou et al. treats MI in unequal probability two-stage setting.
- See also Zhou’s 2014 University of Michigan Ph. D. dissertation.
References I


References III


Thank You!